SHORE

Exam Number: [ ] [ ] [ ]

Year 12
Mathematics Extension 2
HSC Trial Examination
2022

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black or blue pen
- Draw diagrams using pencil
- Board approved calculators may be used
- Answer Questions 1 - 10 on the Multiple Choice answer sheet provided
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Start each of Questions 11 – 16 in a new writing booklet
- Write your examination number on the front cover of each booklet to be handed in
- A NESA reference sheet is provided.
- If you do not attempt a question, submit a blank booklet marked with your examination number and "N/A"

Total marks – 100

Section I Pages 2 - 5

10 marks

- Attempt Questions 1 - 10
- Allow about 15 minutes for this section

Section II Pages 6 - 14

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section.

Note: Any time you have remaining should be spent revising your answers.

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Section I

10 marks

Attempt Questions 1–10

Allow about 15 minutes for this section

Use the Multiple Choice Answer Sheet for Questions 1–10.

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1 Which expression is equal to $\int \frac{1}{x^2+6x+13}\,dx$?

(A) $\frac{1}{\sqrt{13}}\tan^{-1}\left(\frac{x+3}{\sqrt{13}}\right)+C$

(B) $\frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right)+C$

(C) $\frac{1}{4}\ln\left|\frac{x+1}{x+5}\right|+C$

(D) $\frac{1}{4}\ln\left|\frac{x+5}{x+1}\right|+C$

2 What is the angle to the nearest degree subtended at the origin by the points (2, -1, 3) and (-4, 2, 2)?

(A) 12^0

(B) 77^0

(C) 90^0

(D) 103^0

3 The velocity of a particle is given by $v = 3+x$ ms^{-1}. What is the acceleration, in ms^{-2}, of the particle at $x = 2$ m?

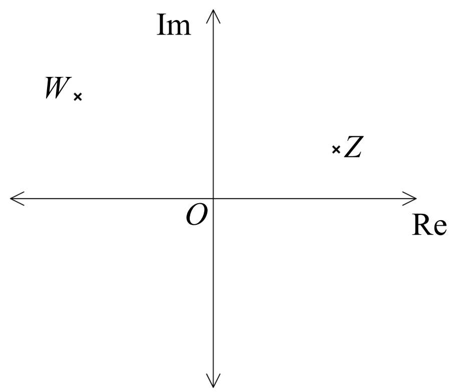
(A) 1

(B) 3

(C) 5

(D) 8

4 The diagram shows the position of points Z and W, that have position vectors z and w respectively. In which quadrant would the point with position vector $\overline{z} - iw$ lie?



(A) 1st

(B) 2nd

(C) 3rd

(D) 4th

5 What is the contrapositive of the statement:

'If Rashford plays and Man United win, I will be a happy chappy'.

(A) If Rashford does not play or Man United do not win, I will not be a happy chappy

(B) If Rashford plays and Man United win, I will not be a happy chappy

(C) If I am a happy chappy, Rashford played and Man United won

(D) If I am not a happy chappy, Rashford did not play or Man United did not win

6 The vector equation of a sphere is given by $\left| \underset{\sim}{v} - \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \right| = 52$. Where does the point with position vector $\begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$ lie with respect to the sphere?

(A) At the centre of the sphere

(B) Within the sphere, but not at the centre

(C) On the surface of the sphere

(D) Outside the sphere.

7 Which of the following statements is true?

(A) $\forall a, b \in \mathbb{R} \quad \sin a = \sin b \Rightarrow a = b$

(B) $\forall a, b \in \mathbb{R} \quad |a+b| > |a-b|$

(C) $\exists a, b \in \mathbb{R}$ such that $\log_e(a+b) = \log_e(ab)$

(D) $\exists a, b \in \mathbb{C} \quad |a+b| > |a| + |b|$

8 Given that $z = 2 - 3i$ is a root of $z^3 - pz^2 + 5z + q = 0$ where p and q are real, what are the values of p and q?

(A) p = -2, q = -26

(B) p = -2, q = 26

(C) p = 2, q = -26

(D) p = 2, q = 26

9 It is given that $f(x)$ is continuous in the interval $-a \le x \le a$, and that $\int_{-a}^{a} f(x)\ dx = 0$.

Which of these does NOT follow?

(A) $\lim_{\varepsilon \to 0} \left(\int_{-a}^{\varepsilon} \frac{dx}{f(x)} + \int_{\varepsilon}^{a} \frac{dx}{f(x)} \right) = 0$

(B) $\int_{-a/2}^{a/2} f(2t)\ dt = 0$

(C) $\int_{c-a}^{a+c} f(x-c)\ \ dx = 0$

(D) $\int_{-a}^{a} f(x) + c\ dx - 2ac = 0$

10 Consider the line segments given by the parametric vector equations below.

$\underset{\sim}{r} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 2 \end{pmatrix}, \quad -2 \le \lambda \le 2;$

$\underset{\sim}{l} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 6 \end{pmatrix}, \quad 0 \le \mu \le 2;$

$\underset{\sim}{v} = \gamma \begin{pmatrix} 9 \\ -6 \end{pmatrix}, \quad 1 \le \gamma \le 3$

Which of the following statements about the line segments is NOT true?

(A) At least two of them are parallel

(B) At least two of them are perpendicular

(C) At least two of them have the same intercept

(D) At least two of them have the same length

End of Section I

Section II

90 marks

Attempt Questions 11 - 16

Allow about 2 hours 45 minutes for this section

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Answer question in a SEPARATE writing booklet.

(a) $z = 2 + i$ and $w = 3 - 4i$. Evaluate $z\overline{w}$. **2**

(b) Given that $a, b \in \mathbb{N}$ and that a is a multiple of 6 and b is a multiple of 4, prove that $2a + 3b$ is a multiple of 12. **2**

(c) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \tan^3 x \sec^2 x \, dx$. **2**

(d) Find $I = \int e^x \sin x \, dx$. **3**

(e) Sketch the curve described by $\dfrac{1}{z} + \dfrac{1}{\overline{z}} = 1$ on the complex plane. **3**

(f) Find the complex roots of $z^3 + 8i = 0$, and show them on an Argand diagram. **3**

Question 12 (15 marks) Answer question in a SEPARATE writing booklet.

(a) Show that the line given by parametric vector equation $\underset{\sim}{r} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ **3**

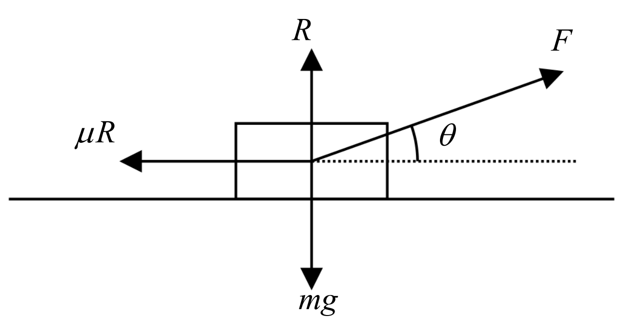
intersects the curve given by $\underset{\sim}{v} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} t \\ t \\ t^2 \end{pmatrix}$ twice.

(b) Prove by contradiction that for rational non-zero *a* and *b* and irrational *x,* that $ax + b$ is irrational. **3**

(c) Use the substitution $t = \tan\frac{\theta}{2}$ to find $\int \frac{1}{1 + \sin\theta + \cos\theta} d\theta$. **3**

(d) Prove that $x^{\frac{5}{2}} + 1 \geq x^2 + x^{\frac{1}{2}}$ for real $x \geq 0$. **3**

(e) An object with mass *m* is being pulled along horizontal ground with acceleration *a* by a constant force *F* acting at angle θ to the horizontal, as shown in the diagram. A force due to gravity of *mg* acts downwards and a normal force *R* acts upwards and a resistive force μR acts parallel to the ground to oppose the motion, where μ, the coefficient of friction, is a constant. **3**



By balancing horizontal and vertical components of forces, show that $\mu = \frac{F\cos\theta - ma}{mg - F\sin\theta}$.

Question 13 (15 marks) Answer question in a SEPARATE writing booklet.

(a) Determine with working whether (4, -1, 5) lies on the line with vector equation **1**

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}.$$

(b) Evaluate: **2**

$$1 + e^{\frac{2\pi i}{3}} + \left(e^{\frac{2\pi i}{3}}\right)^2 + \left(e^{\frac{2\pi i}{3}}\right)^3 + \ldots + \left(e^{\frac{2\pi i}{3}}\right)^{3n}.$$

(c) An object of mass 1 kg is projected vertically upwards from the origin with initial velocity V_0, so that the only forces acting on it are a force due to gravity and air resistance, which is has size $\frac{\dot{y}}{10}$. Taking y as the displacement upwards from the origin, and acceleration due to gravity to be 10 ms^{-2}, its equation of motion is: **3**

$$\ddot{y} = -\left(10 + \frac{\dot{y}}{10}\right)$$

Show that the maximum height the object reaches, $y_{\max}$ is

$$y_{\max} = 10\left(V_0 - 100\ln\left(\frac{100 + V_0}{100}\right)\right).$$

(d) (i) Prove that $e^{in\theta} + e^{-in\theta} = 2\cos n\theta$. **1**

(ii) Use part (i) and the binomial expansion of $\left(e^{i\theta} + e^{-i\theta}\right)^3$ to prove that **2**

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

(iii) Hence find all solutions to $8x^3 - 6x - 1 = 0$. **3**

(e) Given that $u_{n+1} = 2u_n + 3$ and $u_1 = 1$, use mathematical induction to prove that $u_n = 2^{n+1} - 3$ for all positive integers n. **3**

Question 14 (15 marks) Answer question in a SEPARATE writing booklet.

(a) For all natural numbers n, if $(n-5)^2$ is an even integer, prove by contrapositive that n is odd. **2**

(b) Given that $\hat{\underset{\sim}{a}} = x\underset{\sim}{i} + y\underset{\sim}{j} + \frac{\sqrt{7}}{4}k$, $\underset{\sim}{b} = (x-2)\underset{\sim}{i} + y\underset{\sim}{j} + \frac{15}{4\sqrt{7}}\underset{\sim}{k}$ and $\hat{\underset{\sim}{a}} \cdot \underset{\sim}{b} = 0$, determine the value(s) of x and y. **3**

(c) Find $\int \frac{x}{1+\sqrt{x}}\,dx$. **3**

(d) A motorbike that is initially stationary at the origin has mass 400 kg and begins to travel horizontally with a constant propulsive force of 800N. A resistive force opposes the object's motion, proportional to the square of the velocity, such that its equation of motion is:

$$400\ddot{x} = 800 - kv^2$$

Where k is a constant, v is its velocity, and x is displacement in the direction of the motion after t seconds.

(i) Show that $\frac{1}{a^2-b^2} = \frac{1}{2a}\left(\frac{1}{a+b} + \frac{1}{a-b}\right)$. **1**

(ii) Given that the terminal velocity of the motorbike is 40 ms^{-1}, show that the object's velocity is given by **3**

$$v = 40\left(\frac{e^{0.1t}-1}{e^{0.1t}+1}\right).$$

(iii) Find an expression for displacement x in terms of t. **3**

Question 15 (15 marks) Answer question in a SEPARATE writing booklet.

(a) Let $I_n = \int_0^{\frac{\pi}{4}} \cos^n x \, dx$.

(i) Show that $I_n = \frac{1}{n}\left(\frac{1}{\sqrt{2}}\right)^n + \frac{n-1}{n} I_{n-2}$ for $n \geq 2$. **3**

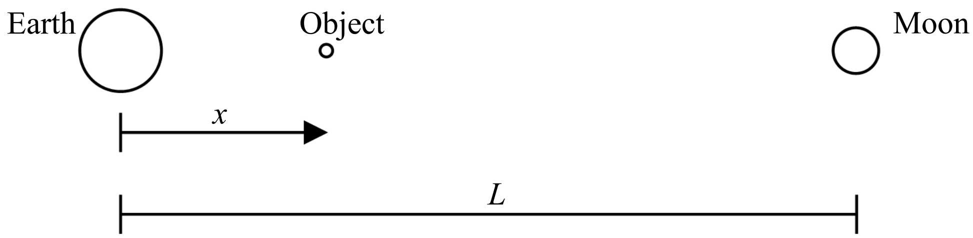
(ii) By using an appropriate trigonometric substitution, show that **2**

$$\int_0^2 \frac{1}{\sqrt{\left(x^2+4\right)^7}} \, dx = \frac{1}{64}\int_0^{\frac{\pi}{4}} \cos^5 \theta \, d\theta .$$

(iii) Hence deduce that $\int_0^2 \frac{1}{\sqrt{\left(x^2+4\right)^7}} \, dx = \frac{43}{3840}\left(\frac{1}{\sqrt{2}}\right)$. **2**

Question 15 continues on page 11

(b) Consider an object travelling from the Earth to the moon. The distance between the centres of the Earth and moon is L and can be assumed to remain constant. The distance of the object from the centre of the Earth in the direction of the moon is x, where $0 < x < L$. Initially the object is at $x = 0.5L$ and is travelling with velocity u ms^{-1}, where $u > 0$.



The equation of motion of the object is given by:

$$\ddot{x} = -\frac{81k}{x^2} + \frac{k}{(L-x)^2}\text{, where } k \text{ is a constant.}$$

(i) Determine that the value of x for which the object has zero acceleration is $x = 0.9L$. **2**

(ii) Show that $v^2 = \frac{162k}{x} + \frac{2k}{L-x} + u^2 - \frac{328k}{L}$. **2**

(iii) By substituting $x = \mu L$ and given that $\frac{u^2 L}{k} = 80$, deduce that the object is stationary at $\mu \approx 0.67$. **3**

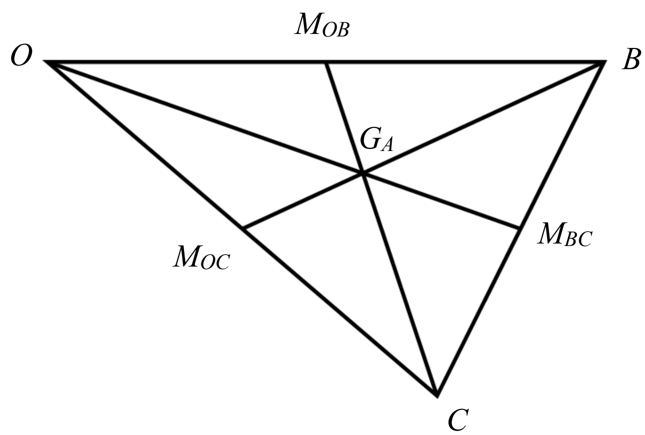
(iv) Briefly describe the object's motion for $t > 0$. **1**

End of Question 15

Question 16 (15 marks) Answer question in a SEPARATE writing booklet.

(a) The lines joining the vertices and the midpoint of the opposite sides of any triangle have a common intersection point called a centroid.

M_{OC} is the midpoint of OC, M_{OB} is the midpoint of OB and M_{BC} is the midpoint of BC. G_A is the centroid. $\overrightarrow{OC}$ is the vector $\underset{\sim}{c}$ and $\overrightarrow{OB}$ is the vector $\underset{\sim}{b}$.

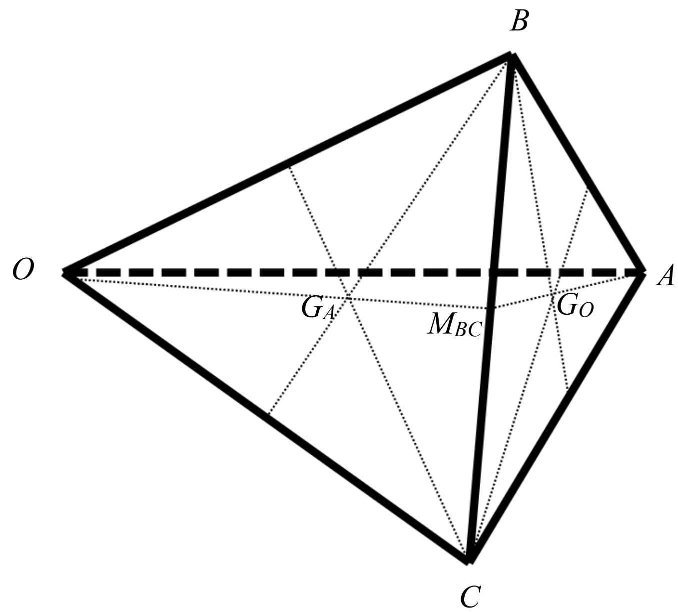


(i) Given that $\overrightarrow{OG_A} = \mu \overrightarrow{OM_{BC}}$, show that $\overrightarrow{OG_A} = \frac{\mu}{2}\left(\underset{\sim}{b} + \underset{\sim}{c}\right)$. **1**

(ii) Show that $\overrightarrow{OG_A} = \frac{2}{3}\overrightarrow{OM_{BC}}$. **2**

Question 16 continues on page 13

A triangular based pyramid has base OAC and apex B as shown.
The centroid of the triangular face opposite to point A is G_A, and similarly for the other centroids. $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$. The midpoint of BC is M_{BC}.



(iii) Show that $\overrightarrow{AG_A} = -\underset{\sim}{a} + \frac{1}{3}\left(\underset{\sim}{b} + \underset{\sim}{c}\right)$. **2**

It is given that: $\overrightarrow{OG_O} = \frac{1}{3}\left(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}\right)$ (Do NOT prove this).

(iv) Show that the line passing through points A and G_A intersects the line passing through points O and G_O. **2**

(v) Find the position vector of the intersection point of the two lines. **1**

b) (i) Use the triangle inequality to show that **2**

$$|z| < \frac{1}{2} \Rightarrow \left|(1+i)z^2 + iz\right| < 1.$$

(ii) Is it true that **1**

$|z| < \frac{1}{2} \Leftrightarrow \left|(1+i)z^2 + iz\right| < 1$? Prove your answer.

Question 16 continues on page 14

c) Suppose that there exists some natural number $n = k$, such that $n^{n+1} > (n+1)^n$.

(i) Use a proof by contradiction to show that this must be true also for $n = k + 1$, by substituting and then multiplying the inequalities. **2**

(ii) Find the smallest value of n for which $n^{n+1} > (n+1)^n$ holds, and then prove the inequality holds for all natural numbers greater than or equal to that value of n. **2**

End of paper

MC1. $\int \frac{1}{x^2+6x+13}\,dx = \int \frac{1}{(x+3)^2+4}\,dx$

$= \frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right)+C$ Ⓑ

2. $\theta = \cos^{-1}\left(\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}\right)$

$= \cos^{-1}\left(\frac{-8-2+6}{\sqrt{14}\sqrt{24}}\right)$

$\approx 103^\circ$ Ⓓ

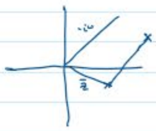
3. $v = 3+x$

$\frac{dv}{dt} = \frac{dx}{dt}$

$= v$

$= 3+x$

@ $x=2$ $a=5$ Ⓒ

4. 

1st Ⓐ

5. Ⓓ

6. Ⓑ (centre @ $(3,0,-2)$, distance of $(-3,0,2)$ from centre is $\sqrt{6^2+4^2}=\sqrt{52}<52$)

7. Ⓒ ($a=b=2$)

8. Roots are α, $2\pm3i$.

$(2+3i)\alpha + (2-3i)\alpha + (2^2+3^2) = 5$

$4\alpha + 13 = 5$

$4\alpha = -8$

$\alpha = -2$

$-2 + (2+3i) + (2-3i) = p$

$p = 2$

$-2(13) = -q$

$q = 26$

Ⓓ

9. Ⓐ (limits do not exist)

10. Ⓒ ($\underset{\sim}{r} \parallel \underset{\sim}{v}$, $\underset{\sim}{t} \perp$ either, $|\underset{\sim}{r}|=|\underset{\sim}{t}|$)

Q11 a) $z = 2+i$, $w = 3-4i$

$z\bar{w} = (2+i)(3+4i)$

$= 6+11i-4$

$= 2+11i$

b) $\exists\, n, k \in \mathbb{Z}$ such that $a=6n$, $b=4k$.

$\therefore\ 2a+3b = 12n+12k$

$= 12(n+k)$

$\therefore\ 2a+3b$ is a multiple of 12.

c) $\int_0^{\frac{\pi}{4}} \tan^3 x \sec^2 x\, dx$

$= \left[\frac{\tan^4 x}{4}\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{4}$

d) $I = \int e^x \sin x \, dx$

$u = \sin x \quad v' = e^x$
$u' = \cos x \quad v = e^x$

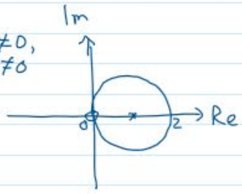
$I = e^x \sin x - \int e^x \cos x \, dx$

$u = \cos x \quad v' = e^x$
$u' = -\sin x \quad v = e^x$

$I = e^x \sin x - \left(e^x \cos x + \int e^x \sin x \, dx\right)$

$2I = e^x(\sin x - \cos x) + 2c$
$I = \frac{1}{2} e^x (\sin x - \cos x) + c$

e) $\frac{1}{z} + \frac{1}{\bar{z}} = 1, \; z \neq 0, \; \bar{z} \neq 0$



$\frac{z + \bar{z}}{z\bar{z}} = 1$

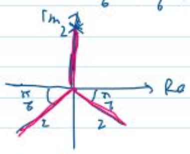
$\frac{2x}{x^2+y^2} = 1$

$2x = x^2 + y^2$
$x^2 - 2x + y^2 = 0$
$(x-1)^2 - 1 + y^2 = 0$
$(x-1)^2 + y^2 = 1$

f) $z^3 + 8i = 0$
$z^3 = -8i$
$r^3 e^{3i\theta} = 2^3 e^{-\frac{i\pi}{2}}$
$r = 2, \quad 3\theta = -\frac{\pi}{2} + 2n\pi$
$\theta = -\frac{\pi}{6} + \frac{2n\pi}{3}$

$\theta = \frac{(4n-1)\pi}{6} \qquad -\pi < \theta \leq \pi$

$\theta = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{2}$



i.e. $z = 2i, 2e^{-\frac{\pi i}{6}}, 2e^{-\frac{5\pi i}{6}}$

12)
a) $\underset{\sim}{r} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$\underset{\sim}{v} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} t \\ t \\ t^2 \end{pmatrix}$

Intersect if they have the same x, y, z components, i.e. if these eqs are consistent:

① $1 + \lambda = 2 + t$
② $-1 + \lambda = t$
③ $1 - \lambda = t^2$
②+③ $0 = t^2 + t$
$t(t+1) = 0$
i.e. $t = 0, \lambda = 1$ or $t = -1, \lambda = 0$ (from ③).

Sub into ①:
LHS $= 1 + 1$
$= 2$
RHS $= 2 + 0$
$=$ LHS

or

LHS $= 1 + 0$
RHS $= 2 + -1$
$=$ LHS

∴ they intersect twice

b) Let $a = \frac{p}{q}$, $b = \frac{r}{t}$ where $p, q, r, t \neq 0 \in \mathbb{Z}$

Assume that $ax + b = \frac{u}{v}$ where $u, v \neq 0$
$u, v \in \mathbb{Z}$

$\therefore \frac{p}{q}x + \frac{r}{t} = \frac{u}{v}$

$ptvx + rqv = uqt$

$x = \frac{uqt - rqv}{ptv}$

Both the numerator and denominator are integers (non-zero integers are a closed set for multiplication and subtraction) $\therefore$ RHS is rational, but x is irrational, so there is a contradiction.
$\therefore ax + b$ is irrational.

c) $\int \frac{1}{1 + \sin\theta + \cos\theta} \, d\theta$

Let $t = \tan\frac{\theta}{2}$

$\frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2}$

$= \frac{1}{2}(1+t^2)$

$d\theta = \frac{2dt}{1+t^2}$

$\int \frac{1}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \frac{2dt}{1+t^2}$

$= \int \frac{2dt}{1+t^2+2t+1-t^2}$

$= \int \frac{2dt}{2+2t} dt$

$= \ln|1 + \tan\frac{\theta}{2}| + c \quad (\theta \neq (2n+1)\pi)$

d) RTP: $x^{5/2} + 1 \geq x^2 + x^{1/2}$

LHS-RHS: $x^{5/2} - x^2 - x^{1/2} + 1$

$x^2(\sqrt{x} - 1) - (\sqrt{x} - 1)$

$(\sqrt{x} - 1)(x^2 - 1)$

Natural domain is $x \geq 0$.

$\therefore$ for $0 \leq x < 1$, $(\sqrt{x} - 1) < 0$ and $(x^2 - 1) < 0$

$\therefore (\sqrt{x} - 1)(x^2 - 1) > 0$.

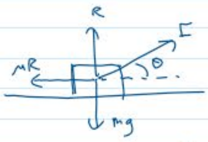
for $x \geq 1$, $(\sqrt{x} - 1) \geq 0$ and $(x^2 - 1) \geq 0$.

$\therefore (\sqrt{x} - 1)(x^2 - 1) \geq 0$

$\therefore$ LHS $-$ RHS ≥ 0 for all real x in domain.

$\therefore x^{5/2} + 1 \geq x^2 + x^{1/2}$

e)



Balancing horizontal forces:

① $F\cos\theta - \mu R = ma \Rightarrow \mu R = F\cos\theta - ma$

Balancing forces vertically:

② $R + F\sin\theta - mg = 0$

From ①, $\mu = \frac{F\cos\theta - ma}{R}$

From ②, $R = mg - F\sin\theta$

$\therefore \mu = \frac{F\cos\theta - ma}{mg - F\sin\theta}$

13. a) For $\lambda = 1$, $\underset{\sim}{s} = \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}$

$\therefore$ $(4, -1, 5)$ does not lie on the line as for $x = 4$ the y & z components don't match.

b) Let $\omega = e^{\frac{2\pi i}{3}}$

$1 + \omega + \omega^2 + \ldots + \omega^{3n}$

$1 + \omega + \omega^2 = 0$

$\therefore$ each sum of 3 consecutive terms is zero.

$\therefore S = (1 + \omega + \omega^2) + \omega^3(1 + \omega + \omega^2) + \ldots + \omega^{3n}$

$= \omega^{3n}$

$= 1$

c) 

$\ddot{y} = -\left(10 + \frac{\dot{y}}{10}\right)$

$\dot{y}\frac{d\dot{y}}{dy} = -\left(10 + \frac{\dot{y}}{10}\right)$

$-\int_{v_0}^{0} \frac{\dot{y}}{10 + \frac{\dot{y}}{10}} \, d\dot{y} = \int_{0}^{y_{max}} dy$

$y_{max} - 0 = 10\int_{0}^{v_0} \frac{\dot{y}}{100 + \dot{y}} \, d\dot{y}$

$y_{max} = 10\int_{0}^{v_0} 1 - \frac{100}{100 + \dot{y}} \, d\dot{y}$

$= 10\left[\dot{y}\right]_0^{v_0} - 1000\left[\ln|100 + \dot{y}|\right]_0^{v_0}$

$= 10v_0 - 1000\ln\left(\frac{100 + v_0}{100}\right)$

$= 10\left(v_0 - 100\ln\left(\frac{100 + v_0}{100}\right)\right)$

d) (i) $e^{in\theta} + e^{-in\theta} = \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)$

(as cosine is even & sine odd)

$= 2\cos(n\theta)$

OR $e^{in\theta} + \overline{e^{in\theta}} = 2\text{Re}(e^{in\theta})$

$= 2\cos(n\theta)$

(ii) $(e^{i\theta} + e^{-i\theta})^3 = e^{3i\theta} + 3e^{i\theta} + 3e^{-i\theta} + e^{-3i\theta}$

$= (e^{3i\theta} + e^{-3i\theta}) + 3(e^{i\theta} + e^{-i\theta})$

$(2\cos\theta)^3 = 2\cos3\theta + 6\cos\theta$

$8\cos^3\theta = 2\cos3\theta + 6\cos\theta$

$\cos3\theta = 4\cos^3\theta - 3\cos\theta$

(iii) $8x^3 - 6x - 1 = 0$

$2(4x^3 - 3x) = 1$

Let $x = \cos\theta$

$2(4\cos^3\theta - 3\cos\theta) = 1$

$2(\cos3\theta) = 1$

$\cos3\theta = \frac{1}{2}$

$3\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

$\theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$

$\therefore$ $x = \cos\frac{\pi}{9}, \cos\frac{5\pi}{9}, \cos\frac{7\pi}{9}$ are solutions.

(Note that the eq^n must have three

(Note that the eq^{n} must have three (complex) roots. Other values of θ that satisfy $\cos 3\theta = \frac{1}{2}$, such as $\frac{11\pi}{9}$, yield the same solution for x $\left(\cos\frac{11\pi}{9} = \cos\frac{7\pi}{9}\right)$)

e) $u_{n+1} = 2u_n + 3$, $u_1 = 1$.

Prove $u_n = 2^{n+1} - 3$.

For $n=1$, $u_1 = 1$

and $u_1 = 2^{1+1} - 3$

$= 4 - 3$

$= 1$

$\therefore$ true for $n=1$

Assume true for $n=k$,

i.e. $u_k = 2^{k+1} - 3$. $(*)$

RTP true for $n=k+1$,

i.e. $u_{k+1} = 2^{k+2} - 3$

$u_{k+1} = 2u_k + 3$ (given)

$= 2(2^{k+1} - 3) + 3$ (using $*$)

$= 2^{k+2} - 6 + 3$

$= 2^{k+2} - 3$

$\therefore$ true for $n=k+1$.

$\therefore$ true for all positive integers n by mathematical induction

14 a) Contrapositive: RTP:

'If n is even then $(n-5)^2$ is odd.'

$\exists k \in \mathbb{Z} \mid n = 2k$.

$(2k-5)^2 = 4k^2 - 20k + 25$

$= 2(2k^2 - 10k + 12) + 1$

$\therefore (2k-5)^2$ is odd.

$\therefore$ contrapositive statement is true

$\therefore$ if $(n-5)^2$ is even, n is odd.

b) $|\hat{a}| = 1$ by definition.

$\therefore x^2 + y^2 + \frac{7}{16} = 1$

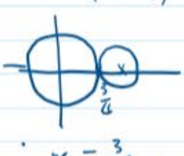
$x^2 + y^2 = \frac{9}{16}$ ①

Also, $\hat{a} \cdot b = 0$

$x^2 - 2x + y^2 + \frac{15}{16} = 0$

$(x-1)^2 - 1 + y^2 + \frac{15}{16} = 0$

$(x-1)^2 + y^2 = \frac{1}{16}$



$\therefore x = \frac{3}{4}$, $y = 0$

OR ① − ②: $2x - \frac{15}{16} = \frac{9}{16}$

$2x = \frac{24}{16} = \frac{3}{2}$

$x = \frac{3}{4}$

Sub into ①: $\left(\frac{3}{4}\right)^2 + y^2 = \frac{9}{16}$

$y^2 = 0$, $y = 0$ $\therefore x = \frac{3}{4}$, $y = 0$

d $\int_0^1 \frac{x}{1+\sqrt{x}}\,dx$ Let $u = 1+\sqrt{x}$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$dx = 2\sqrt{x}\,du$

$dx = 2(u-1)\,du$

$x = (u-1)^2$

$\int \frac{(u-1)^2}{u} 2(u-1)\,du$

$= 2\int \frac{(u-1)^3}{u}\,du$

$= 2\int \frac{u^3-3u^2+3u-1}{u}\,du$

$= 2\int u^2 - 3u + 3 - \frac{1}{u}\,du$

$= 2\left(\frac{u^3}{3} - \frac{3u^2}{2} + 3u - \ln|u|\right) + C$

$= \frac{2(1+\sqrt{x})^3}{3} - 3(1+\sqrt{x})^2 + 6(1+\sqrt{x}) - \ln(1+\sqrt{x}) + C$

OR

$\int \frac{x}{1+\sqrt{x}}\,dx$ Let $u = \sqrt{x}$ $(x \geq 0)$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$dx = 2u\,du$

$\int \frac{u^2}{1+u} 2u\,du$

$= 2\int \frac{u^3}{1+u}\,du$

$= 2\int \frac{u^2(u+1) - u(u+1) + (u+1) - 1}{u+1}\,du$

$= 2\int u^2 - u + 1 - \frac{1}{u+1}\,du$

$= 2\frac{u^3}{3} - u^2 + 2u - 2\ln(u+1) + C$

$= \frac{2x\sqrt{x}}{3} - x + 2\sqrt{x} - 2\ln(\sqrt{x}+1) + C$

OR

$\int \frac{x}{1+\sqrt{x}}\,dx$ Let $x = \tan^4 u$

$\frac{dx}{du} = 4\tan^3 u \sec^2 u$

$4\int \frac{\tan^4 u}{1+\tan^2 u} \tan^3 u \sec^2 u\,du$ (note that $|\tan^2 u| = \tan^2 u$)

$= 4\int \frac{\tan^7 u}{\sec^2 u} \sec^2 u\,du$ (as $1+\tan^2 u = \sec^2 u$)

$= 4\int \tan^5 u \sec^2 u - \tan^5 u\,du$

$= 4\int \tan^5 u \sec^2 u - \tan^3 u \sec^2 u + \tan^3 u\,du$

$= 4\int \tan^5 u \sec^2 u - \tan^3 u \sec^2 u + \tan u \sec^2 u - \tan u\,du$

$= 4\left(\frac{\tan^6 u}{6} - \frac{\tan^4 u}{4} + \frac{\tan^2 u}{2} + \ln(\cos u)\right) + C$

$= \frac{2}{3}x^{3/2} - x + 2\sqrt{x} + 4\ln\left(\frac{1}{\sqrt{1+\sqrt{x}}}\right) + C$

$\left(= \frac{2}{3}x^{3/2} - x + 2\sqrt{x} - 2\ln(1+\sqrt{x}) + C\right)$

$\left(\cos u = \frac{1}{\sec u} = \frac{1}{\sqrt{1+\tan^2 u}} = \frac{1}{\sqrt{1+\sqrt{x}}} = (1+\sqrt{x})^{-\frac{1}{2}}\right)$

d) $400\ddot{x} = 800 - kv^2$

(i) $\frac{1}{2a}\left(\frac{1}{a+b} + \frac{1}{a-b}\right) = \frac{1}{2a}\left(\frac{a-b+a+b}{(a+b)(a-b)}\right)$

$= \frac{1}{2a}\left(\frac{2a}{a^2-b^2}\right)$

$= \frac{1}{a^2-b^2}$

(ii) $v_T = 40$

@ $\ddot{x} = 0$,

$0 = 800 - k(40)^2$

$k = \frac{800}{40^2}$

$k = \frac{1}{2}$

$400\frac{dv}{dt} = 800 - \frac{1}{2}v^2$

$\int_0^v \frac{400}{800 - \frac{1}{2}v^2}\,dv = \int_0^t dt$

$800\int_0^v \frac{1}{1600 - v^2}\,dv = t - 0$

$\frac{800}{2(40)}\int_0^v \frac{1}{40+v} + \frac{1}{40-v}\,dv = t$

$10\left[\ln\left(\frac{40+v}{40-v}\right)\right]_0^v = t$

$\ln\left(\frac{40+v}{40-v}\right) = 0.1t$

$40 + v = e^{0.1t}(40 - v)$

$v(e^{0.1t} + 1) = 40(e^{0.1t} - 1)$

$v = 40\left(\frac{e^{0.1t} - 1}{e^{0.1t} + 1}\right)$

(iii) $\frac{dx}{dt} = 40\left(\frac{e^{0.1t} - 1}{e^{0.1t} + 1}\right)$

$\int_0^x dx = 40\int_0^t \frac{e^{0.1t} - 1}{e^{0.1t} + 1}\,dt$

$x - 0 = 40\int_0^t 1 - \frac{2}{e^{0.1t} + 1}\,dt$

$x = 40\int_0^t 1 + \left(\frac{2}{0.1}\right)\frac{(-0.1)e^{-0.1t}}{1 + e^{-0.1t}}\,dt$

$= 40\left[t + 20\ln(1 + e^{-0.1t})\right]_0^t$

$= 40t + 800\ln\left(\frac{1 + e^{-0.1t}}{2}\right)$

OR

$x = 800\ln\left(\frac{1 + e^{0.1t}}{2}\right) - 40t$

OR

$x = 400\ln\left(\frac{(e^{0.1t} + 1)^2}{4e^{0.1t}}\right)$

OR

$x = 400\ln\left(\frac{(e^{0.1t} + 1)(e^{-0.1t} + 1)}{4}\right)$

OR

$x = 800\ln\left(\frac{e^{0.05t} + e^{-0.05t}}{2}\right)$

15.a) $I_n = \int_0^{\frac{\pi}{4}} \cos^n x \, dx$

(i) RTP $I_n = \frac{1}{n}\left(\frac{1}{\sqrt{2}}\right)^n + \frac{n-1}{n}\int_0^{\frac{\pi}{4}} \cos^{n-2} x \, dx$

$u = \cos^{n-1} x \qquad v' = \cos x$

$u' = -(n-1)\sin x \cos^{n-2} x \qquad v = \sin x$

$I_n = \left[\sin x \cos^{n-1} x\right]_0^{\frac{\pi}{4}} + (n-1)\int_0^{\frac{\pi}{4}} \sin^2 x \cos^{n-2} x \, dx$

$= \left(\frac{1}{\sqrt{2}}\right)^n + (n-1)\int_0^{\frac{\pi}{4}} (1-\cos^2 x)\cos^{n-2} x \, dx$

$= \left(\frac{1}{\sqrt{2}}\right)^n + (n-1)\left[I_{n-2} - I_n\right]$

$I_n(1+n-1) = \left(\frac{1}{\sqrt{2}}\right)^n + (n-1) I_{n-2}$

$I_n = \frac{1}{n}\left(\frac{1}{\sqrt{2}}\right)^n + \frac{n-1}{n} I_{n-2}$

(ii) $\int_0^2 \frac{1}{\sqrt{(x^2+4)^7}} \, dx$

Let $\theta = \tan^{-1}\left(\frac{x}{2}\right)$

$x = 2\tan\theta$

$\frac{dx}{d\theta} = 2\sec^2\theta$

@ $x = 2, \theta = \frac{\pi}{4}$

$x = 0, \theta = 0$

$\int_0^{\frac{\pi}{4}} \frac{2\sec^2\theta}{\sqrt{[4(\tan^2\theta+1)]^7}} \, d\theta$

$= \frac{2}{4^{\frac{7}{2}}}\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{|\sec^7\theta|} \, d\theta$ $\qquad \sec\theta > 0$ for $0 \le \theta \le \frac{\pi}{4}$

$= \frac{1}{64}\int_0^{\frac{\pi}{4}} \cos^5\theta \, d\theta$

(iii) $\int_0^2 \frac{1}{\sqrt{(x^2+4)^7}} \, dx = \frac{I_5}{64}$

$I_1 = \int_0^{\frac{\pi}{4}} \cos\theta \, d\theta$

$= \sin\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}}$

$I_3 = \frac{1}{3}\left(\frac{1}{\sqrt{2}}\right)^3 + \frac{2}{3}\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{1}{6}\left(\frac{1}{\sqrt{2}}\right) + \frac{4}{6}\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{5}{6}\left(\frac{1}{\sqrt{2}}\right)$

$I_5 = \frac{1}{5}\left(\frac{1}{\sqrt{2}}\right)^5 + \frac{4}{5}\left(\frac{5}{6}\left(\frac{1}{\sqrt{2}}\right)\right)$

$= \frac{1}{20}\left(\frac{1}{\sqrt{2}}\right) + \frac{2}{3}\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{23}{60}\left(\frac{1}{\sqrt{2}}\right)$

$\therefore \int_0^2 \frac{1}{\sqrt{(x^2+4)^7}} \, dx = \frac{1}{64}\left(\frac{23}{60}\right)\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{23}{3840}\left(\frac{1}{\sqrt{2}}\right)$

b) (i) Solve $\ddot{x} = 0$:

$$0 = \frac{-81k}{x^2} + \frac{k}{(L-x)^2}$$

$$\frac{81}{x^2} = \frac{1}{(L-x)^2}$$

$$\frac{x^2}{81} = (L-x)^2$$

$$\frac{x}{9} = \pm(L-x)$$

$x = 9L - 9x$ | $x = 9x - 9L$
$10x = 9L$ | $8x = 9L$
$x = \frac{9}{10}L$ | $x = \frac{9}{8}L$
but $0 < x < L$

$\therefore x = \frac{9}{10}L$
$= 0.9L$

OR

$$0 = -81(L-x)^2 + x^2$$

$$0 = -81x^2 + 162Lx - 81L^2 + x^2$$

$$80x^2 - 162Lx + 81L^2 = 0$$

$$(10x - 9L)(8x - 9L) = 0$$

$x = \frac{9}{10}L$ or $x = \frac{9}{8}L$ but $x < L$

$\therefore x = \frac{9}{10}L$.

(ii) $$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -81kx^{-2} + k(L-x)^{-2}$$

$$\frac{1}{2}v^2 = \frac{81k}{x} + \frac{k}{L-x} + \frac{c}{2}$$

$$v^2 = \frac{162k}{x} + \frac{2k}{L-x} + c$$

@ $t=0$, $x = \frac{1}{2}L$ $v = u$:

$$u^2 = \frac{324k}{L} + \frac{4k}{L} + c$$

$$c = u^2 - \frac{328k}{L}$$

$$\therefore v^2 = \frac{162k}{x} + \frac{2k}{L-x} + u^2 - \frac{328k}{L}$$

(iii) @ $v=0$:

$$0 = \frac{162k}{\mu L} + \frac{2k}{L(1-\mu)} + u^2 - \frac{328k}{L}$$

$$0 = 162(1-\mu) + 2\mu + \frac{u^2L}{k}\mu(1-\mu) - 328\mu(1-\mu)$$

$$= 162 - 162\mu + 2\mu + (80 - 328)\mu(1-\mu)$$

$$0 = 162 - 160\mu - 248\mu + 248\mu^2$$

$$248\mu^2 - 408\mu + 162 = 0$$

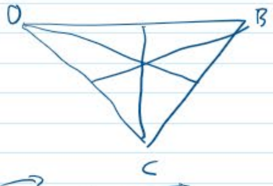
$$\mu = \frac{408 \pm \sqrt{408^2 - 4(248)(162)}}{2 \times 248}$$

$$\mu \approx 0.98, \; 0.67$$

i.e. $v = 0$ @ $\mu \approx 0.67$

(iv) The object slows down until it comes to a stop @ $x \approx 0.67L$, it then changes direction and accelerates towards Earth.

16. a) (i)



$$\overrightarrow{OG_A} = \mu\,\overrightarrow{OM_{BC}}$$
$$= \mu(\overrightarrow{OB} + \tfrac{1}{2}\overrightarrow{BC})$$
$$= \mu(\underset{\sim}{b} + \tfrac{1}{2}(\underset{\sim}{c} - \underset{\sim}{b}))$$
$$= \mu(\underset{\sim}{b} + \tfrac{1}{2}\underset{\sim}{c} - \tfrac{1}{2}\underset{\sim}{b})$$
$$= \frac{\mu}{2}(\underset{\sim}{b} + \underset{\sim}{c})$$

(ii) $$\overrightarrow{OG_A} = \overrightarrow{OC} + \lambda\,\overrightarrow{CM_{OB}}$$
$$= \underset{\sim}{c} + \lambda(-\underset{\sim}{c} + \tfrac{1}{2}\underset{\sim}{b})$$

Equate coeffs of (basis vectors) $\underset{\sim}{b}$ & $\underset{\sim}{c}$.

$\underset{\sim}{b}$: $\frac{\mu}{2} = \frac{\lambda}{2} \Rightarrow \mu = \lambda$

$\underset{\sim}{c}$: $\frac{\mu}{2} = 1 - \lambda$

Sub $\lambda = \mu$:

$$\frac{\mu}{2} = 1 - \mu$$
$$\mu = 2 - 2\mu$$
$$3\mu = 2$$
$$\mu = \frac{2}{3}$$

$\therefore \overrightarrow{OG_A} = \frac{2}{3}\overrightarrow{OM_{BC}}$

(iii) $$\overrightarrow{AG_A} = \overrightarrow{AO} + \overrightarrow{OG_A}$$
$$= -\underset{\sim}{a} + \tfrac{2}{3}\overrightarrow{OM_{BC}}$$
$$= -\underset{\sim}{a} + \tfrac{2}{3}(\overrightarrow{OB} + \tfrac{1}{2}\overrightarrow{BC})$$
$$= -\underset{\sim}{a} + \tfrac{2}{3}(\underset{\sim}{b} + \tfrac{1}{2}(\underset{\sim}{c} - \underset{\sim}{b}))$$
$$= -\underset{\sim}{a} + \tfrac{1}{3}(\underset{\sim}{b} + \underset{\sim}{c})$$

(iv) $$\underset{\sim}{r}_A = \underset{\sim}{a} + \lambda\,\overrightarrow{AG_A}$$
$$= \underset{\sim}{a} + \lambda(-\underset{\sim}{a} + \tfrac{1}{3}(\underset{\sim}{b} + \underset{\sim}{c}))$$

$$\underset{\sim}{r}_o = \frac{\mu}{3}(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c})$$

Equate coeffs of basis vectors:

$$1 - \lambda = \frac{\mu}{3}$$
$$\tfrac{1}{3}\lambda = \tfrac{1}{3}\mu$$
$$\tfrac{1}{3}\lambda = \tfrac{1}{3}\mu$$

Only two independent eq^{n}s therefore lines intersect:

$$\lambda = \mu$$
$$1 - \lambda = \frac{\lambda}{3}$$
$$1 = \frac{4\lambda}{3}$$
$$\lambda = \frac{3}{4}.$$

(v) @ $\lambda = \frac{3}{4}$,

$$\underset{\sim}{r}_A = \underset{\sim}{a}(1 - \tfrac{3}{4}) + \tfrac{1}{4}\underset{\sim}{b} + \tfrac{1}{4}\underset{\sim}{c}$$

i.e. centroid of pyramid has position vector $\underset{\sim}{r} = \frac{1}{4}(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c})$

b) (i) $|(1+i)z^2 + iz| < |(1+i)z^2| + |iz|$ } triangle inequality

$< |z|^2 + |z|^2 + |z|$

$< \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \frac{1}{2}$ as $|z| < \frac{1}{2}$

$= 1$

$\therefore |(1+i)z^2 + iz| < 1$

(ii) Substitute $z = -\frac{1}{2}$:

$\left|\frac{1}{4} - \frac{1}{4}i\right| = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2}$

$= \frac{1}{16}\sqrt{2}$

< 1 BUT $|z| \geq \frac{1}{2}$

$\therefore |(1+i)z^2 + iz| < 1 \not\Rightarrow |z| < \frac{1}{2}$ by counterexample

c) (i) $k^{k+1} > (k+1)^k$

Assume not true for $n = k+1$,

i.e. $(k+1)^{k+2} \leq (k+2)^{k+1}$ ①

$(k+1)^k < k^{k+1}$ ②

①×②: $(k+1)^{2k+2} < k^{k+1}(k+2)^{k+1}$

$((k+1)^2)^{k+1} < (k^2+2k)^{k+1}$

$(k^2+2k+1)^{k+1} < (k^2+2k)^{k+1}$

Not true as $k^2+2k+1 > k^2+2k$.

$\therefore$ true for $n = k+1$,

i.e. $(k+1)^{k+2} > (k+2)^{k+1}$ if $k^{k+1} > (k+1)^k$

(ii) For $n=1$, $1^2 < 2^1$

$n=2$, $2^3 < 3^2$

$n=3$, $3^4 > 4^3$

i.e. true for $n=3$.

Assume true for $n=k$,

i.e. $k^{k+1} > (k+1)^k$

$\therefore$ true for $n=k+1$,

i.e. $(k+1)^{k+2} > (k+2)^{k+1}$ from (i)

$\therefore$ true for all $n \geq 3$ by mathematical induction.